Lecture 24

Principle of Inclusion-Exclusion (contd.) and its applications

Principle of Inclusion-Exclusion

Theorem: Let $A_1, A_2, \ldots, A_n$ be the finite sets. Then,

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_{j=1}^{n} (-1)^{j-1} \sum_{i_1, i_2, \ldots, i_j} |A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_j}|$$

Proof: We show that each element in $A_1 \cup A_2 \cup \ldots \cup A_n$ is counted exactly once on the RHS.

Let $x \in A_1 \cup A_2 \cup \ldots \cup A_n$.

Let $S = \{i_1, i_2, \ldots, i_s\}$ such that $x \in A_i$ iff $i \in S$. Let $s = |S|$.

Observation: $A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}$ contains x if and only if $\{i_1, i_2, \ldots, i_k\} \subseteq S$.

Principle of Inclusion-Exclusion

x is counted or subtracted the following times on the RHS:

- Counted s times in single intersections.
- Subtracted $\binom{s}{2}$ times in double intersections.
- Counted $\binom{s}{3}$ times in triple intersections.

$\vdots$

Follows from the alternating addition and subtraction of binomial coefficients.

So, RHS counts x

$$s - \binom{s}{2} + \binom{s}{3} - \binom{s}{4} \cdots + (-1)^{s-1}\binom{s}{s} = 1$$

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Derangemenets: Application of PIE

Definition: A permutation of $[n]$ that has no fixed points, i.e., no i is in the ith position, is called a derangement of $[n]$.

Examples: $23451, 21453, 54231, 41523$ are some derangements of $[5]$.

$24315, 23145, 51342, 32451$ are some permutations of $[5]$ which are not derangements.

Theorem: The number of derangements of $[n]$ is, $D(n) = \sum_{i=0}^{n} (-1)^i \frac{n!}{i!}$

Proof: # derangements of $[n] = n! -$ # permutations of $[n]$ with at least one fixed point

Let A_i denote the set of permutations of $[n]$ in which element i is in the ith position.

We want to find $|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_{j=1}^{n} (-1)^{j-1} \sum_{i_1, i_2, \ldots, i_j} |A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_j}|$

Derangemenets: Application of PIE

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_{j=1}^{n} (-1)^{j-1} \sum_{i_1, i_2, \ldots, i_j} |A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_j}|$$

Contributions on the RHS from:

- Single intersections: $\sum_i |A_i| = n\,.\,(n-1)! = n!$
- Double intersections: $-\sum_{i_1, i_2} |A_{i_1} \cap A_{i_2}| = -\binom{n}{2}(n-2)! = -\frac{n!}{2!}$

$\vdots$

- j intersections: $(-1)^{j-1} \sum_{i_1, i_2, \ldots, i_j} |A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_j}| = (-1)^{j-1}\binom{n}{j}(n-j)! = (-1)^{j-1}\frac{n!}{j!}$

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = n! - \frac{n!}{2!} + \frac{n!}{3!} - \ldots + (-1)^{n-1}\frac{n!}{n!}$$

Derangemenets: Application of PIE

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = n! - \frac{n!}{2!} + \frac{n!}{3!} - \ldots + (-1)^{n-1}\frac{n!}{n!}$$

Now,

$$\begin{aligned} D(n) &= n! - |A_1 \cup A_2 \cup \ldots \cup A_n| \\ &= n! - n! + \frac{n!}{2!} - \frac{n!}{3!} + \ldots + (-1)^n \frac{n!}{n!} \\ &= \sum_{i=0}^{n} (-1)^i \frac{n!}{i!} \end{aligned}$$

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Counting Surjections: Application of PIE

Let n and k be positive integers such that $n \geq k$.

How many surjections are there from $[n]$ to $[k]$?

We want to find out this

$$\# \text{ surjections from } [n] \text{ to } [k] = k^n - \# \text{ functions that are not surjections from } [n] \text{ to } [k]$$

Total # functions from $[n]$ to $[k]$.

Observation: A non-surjective function misses at least one element in the range.

Let A_i be the set of functions from $[n]$ to $[k]$ whose range does not contain i.

We want to find $|A_1 \cup A_2 \cup \ldots \cup A_k| = \sum_{j=1}^{k} (-1)^{j-1} \sum_{i_1, i_2, \ldots, i_j} |A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_j}|$

Counting Surjections: Application of PIE

$$|A_1 \cup A_2 \cup \ldots \cup A_k| = \sum_{j=1}^{k} (-1)^{j-1} \sum_{i_1, i_2, \ldots, i_j} |A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_j}|$$

Contributions on the RHS from:

- Single intersections: $\sum_{i} |A_i| = k \,.\, (k-1)^n$
- Double intersections: $-\sum_{i_1, i_2} |A_{i_1} \cap A_{i_2}| = -\binom{k}{2}(k-2)^n$

$\vdots$

- j intersections: $(-1)^{j-1} \sum_{i_1, i_2, \ldots, i_j} |A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_j}| = (-1)^{j-1}\binom{k}{j}(k-j)^n$

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_{j=1}^{k} (-1)^{j-1} \binom{k}{j}(k-j)^n$$

← *Subtract this from k^n to get # surjections.*